**Properties.** To get a better understanding of our intermediacy measure, we study some of its properties. We first consider the properties of path addition and path contraction. We show that both adding paths and contracting paths lead to an increase in intermediacy.

We start by considering the property of path addition. We define path addition as follows.

**Definition.** Consider a directed acyclic graph G = (V, E) and two nodes u, v \in V such that there does not exist a path from node v to node u. *Path addition* is the operation in which a new path from node u to node v is added. Let l denote the length of the new path. If l = 1, an edge (u, v) is added. If l > 1, nodes w\_1, \ldots, w\_{l - 1} and edges (u, w\_1), (w\_1, w\_2), \ldots, (w\_{l - 2}, w\_{l - 1}), (w\_{l - 1}, v) are added.

This definition includes the condition that there does not exist a path from node v to node u. Please note that this condition is needed to ensure that graph G will remain acyclic after adding paths. The following theorem states that adding paths increases intermediacy.

**Theorem.** Consider a directed acyclic graph G = (V, E), a source s \in V, and a target t \in V. In addition, consider two nodes u, v \in V such that there does not exist a path from node v to node u. Adding a path from node u to node v increases the intermediacy \phi\_w of any node w located on a path from the source s to node u or from node v to the target t.

*Proof.* ???.

We note that the theorem does not depend on the parameter p. Hence, adding paths always increases intermediacy, regardless of the value of p. To illustrate the theorem, consider Figs. 2A and 2B. The graph in Fig. 2B is identical to the one in Fig. 2A except that two paths from node u to node v have been added. As can be seen, adding these paths increases the intermediacy of nodes located between the source s and node u or between node v and the target t, including nodes u and v themselves. This reflects the basic intuition that, after paths from node u to node v have been added, going from the source s to the target t through nodes u and v has become ‘easier’ than it was before. This means that nodes located between the source s and node u or between node v and the target t have become more important in connecting the source and the target. Consequently, the intermediacy of these nodes has increased.

We now consider the property of path contraction. We use P\_{uv} to denote the set of all nodes located on a path from a node u to a node v, including nodes u and v themselves. Path contraction is then defined as follows.

**Definition.** Consider a directed acyclic graph G = (V, E) and two nodes u, v \in V such that there exists at least one path from node u to node v. *Path contraction* is the operation in which all nodes in P\_{uv} are contracted. This means that the nodes in P\_{uv} are replaced by a new node w. Edges pointing from a node v’ \notin P\_{uv} to a node in P\_{uv} are replaced by a new edge (v’, w). Edges pointing from a node in P\_{uv} to a node v’ \notin P\_{uv} are replaced by a new edge (w, v’). Edges between nodes in P\_{uv} are removed.

The following theorem states that contracting paths increases intermediacy.

**Theorem.** Consider a directed acyclic graph G = (V, E), a source s \in V, and a target t \in V. In addition, consider two nodes u, v \in V such that there exists at least one path from node u to node v and such that nodes in P\_{uv} do not have neighbors outside P\_{uv} except for incoming neighbors of node u and outgoing neighbors of node v. Contracting paths from node u to node v increases the intermediacy \phi\_w of any node w located on a path from the source s to node u or from node v to the target t.

*Proof.* ???.

Like Theorem ???, Theorem ??? does not depend on the parameter p. Regardless of the value of p, contracting paths always increases intermediacy. Theorem ??? is illustrated in Figs. 2B and 2C. The graph in Fig. 2C is identical to the one in Fig. 2B except that paths from node u to node v have been contracted. As a result, there has been an increase in the intermediacy of nodes located between the source s and node u or between node v and the target t, including nodes u and v themselves (which have been contracted into a single node). This reflects the basic intuition that, after paths from node u to node v have been contracted, going from the source s to the target t through nodes u and v has become ‘easier’ than it was before. In other words, nodes located on a path from the source s to the target t going through nodes u and v have become more important in connecting the source and the target, and hence the intermediacy of these nodes has increased.

**Comparison to alternative approaches.** What distinguishes intermediacy from alternative approaches? Consider the graph shown in Fig. 3A. To get from the source s to the target t, one could either take a path going through nodes u and v or one could take the path going through node w. Based on intermediacy, the latter path represents a stronger connection between the source and the target than the former one. This follows directly from the path contraction property.

Interestingly, main path analysis gives the opposite result. This can be seen in Fig. 3B. For each edge, the figure shows the search path count, which is the number of source-target paths that make use of that edge. Because the search path counts of (s, u) and (v, t) are higher than the search path counts of (s, w) and (w, t), main path analysis favors paths going through nodes u and v over the path going through node w. This is exactly opposite to the result obtained using intermediacy. Fig. 3B makes clear that main path analysis yields outcomes that violate the path contraction property. Main path analysis may favor longer paths over shorter ones. We consider this behavior to be counterintuitive. We note that there are various variants of main path analysis. In Fig. 3B, the variant based on search path counts is used. Other variants of main path analysis suffer from similar counterintuitive behavior.

The intermediacy of a node v is defined as the probability that there exists at least one active source-target path that goes through node v. As an alternative, one could consider defining the intermediacy of a node v as the expected number of active source-target paths that go through node v. This alternative definition of intermediacy is illustrated in Fig. 3C. In this figure, nodes u and v have a higher intermediacy than node w. Paths going through nodes u and v may therefore be favored over the path going through node w. Fig. 3C shows that the alternative definition of intermediacy does not have the path contraction property. Using the alternative definition, depending on the value of the parameter p, contracting paths may decrease rather than increase intermediacy. Because we consider this to be counterintuitive, we do not use the alternative definition of intermediacy.

Fig. 2. Illustration of the properties of path addition and path contraction. Comparing (B) to (A) shows how path addition increases intermediacy. Comparing (C) to (B) shows how path contraction increases intermediacy. The parameter p equals ???.

Fig 3. Comparison of intermediacy (A) to main path analysis (B) and to an alternative definition of intermediacy (C). The parameter p equals ???.